

# Swapping algebra, Virasoro algebra and discrete integrable system

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*To William Goldman on the occasion of his sixtieth birthday*

the date of receipt and acceptance should be inserted later

**Abstract** We induce a Poisson algebra  $\{\cdot, \cdot\}_{\mathcal{C}_{n,N}}$  on the configuration space  $\mathcal{C}_{n,N}$  of  $N$  twisted polygons in  $\mathbb{RP}^{n-1}$  from the swapping algebra [L12], which is found coincide with Faddeev-Takhtajan-Volkov algebra for  $n = 2$ . There is another Poisson algebra  $\{\cdot, \cdot\}_{S_2}$  on  $\mathcal{C}_{2,N}$  induced from the first Adler-Gelfand-Dickey Poisson algebra by Miura transformation. By observing that these two Poisson algebras are asymptotically related to the dual to the Virasoro algebra, finally, we prove that  $\{\cdot, \cdot\}_{\mathcal{C}_{2,N}}$  and  $\{\cdot, \cdot\}_{S_2}$  are Schouten commute.

**Keywords** Swapping algebra · configuration space of  $N$  twisted polygons · Virasoro algebra · Poisson-Lie group · bihamiltonian.

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## 1 Introduction

The space  $\mathcal{L}_n$  of ordinary differential operators of the form  $\partial_n + u_2\partial_{n-2} + \dots + u_n$  has two Poisson structures, called the first(second resp.) Adler-Gelfand-Dickey Poisson structure [A79] [GD87]. These two Poisson structures are realized by Drinfeld-Sokolov reductions [DS85] via the hamiltonian reductions of the dual space to the affine Kac-Moody algebra  $\widehat{\mathfrak{gl}(n, \mathbb{R})}$ . As a consequence, these two Poisson structures are compatible and provide a bihamiltonian system. We view the configuration space  $\mathcal{C}_{n,N}$  of  $N$ -twisted polygons in  $\mathbb{RP}^{n-1}$  as the discrete version of the space  $\mathcal{L}_n$ . For  $n = 3$ , R. Schwartz, V. Ovsienko and S. Tabachnikov [SOT10] introduced the Poisson structure  $\{\cdot, \cdot\}_{S_3}$  corresponding to the first Adler-Gelfand-Dickey Poisson structure defined on the coordinate function ring of  $\mathcal{C}_{3,N}$ . They show that the pentagram map relative to this Poisson structure is completely integrable. They conjecture that there exists a Poisson structure  $\{\cdot, \cdot\}_{C_3}$  on the coordinate function ring of  $\mathcal{C}_{3,N}$  corresponding to the second Adler-Gelfand-Dickey Poisson structure(it is also mentioned by B. Khsein, F. Soloviev in [KS13] for  $n$  in general and many others). Moreover,

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they conjecture that  $\{\cdot, \cdot\}_{C3}$  is compatible with their Poisson structure. It is very interesting to consider these two conjectures for  $n$  in general. But the Poisson structure corresponding to the first Adler-Gelfand-Dickey Poisson bracket on  $\mathcal{C}_{n,N}$  for  $n > 3$  is still missing. The latter conjecture is not clear even for  $n = 2$ .

The swapping algebra [L12], introduced by F. Labourie, is used to characterize the Atiyah-Bott-Goldman Poisson structure [AB83][G84] for the Hitchin component [H92] and the second Adler-Gelfand-Dickey Poisson structure for the space of realopers. Taken linearity relations into consideration, the quotient algebra—rank  $n$  swapping algebra [Su16] is used to characterize the Fock–Goncharov Poisson structure [FG06] on cluster  $\mathcal{X}_{\text{PSL}(n, \mathbb{R}), S}$  moduli space in [Su15]. In this paper, for  $n \geq 2$ , we define the Poisson structures on the coordinate function ring of  $\mathcal{C}_{n,N}$  induced from the swapping algebra [L12], the induced Poisson structures are corresponding to the second Adler-Gelfand-Dickey Poisson structures. For  $n = 2$ , the Poisson algebra induced from the swapping algebra is the same as Takhtajan-Faddeev-Volkov algebra, which is studied by L. A. Takhtajan, L. D. Faddeev, A. Yu. Volkov in 90s [FT86] [FV93] [V88] [V92] and many others in lattice Virasoro algebra.

We consider a Poisson bracket  $\{\cdot, \cdot\}_{S2}$  for  $\mathcal{C}_{2,N}$ , which is a “degenerate” version of the R. Schwartz et al.’s Poisson bracket  $\{\cdot, \cdot\}_{S3}$  for  $\mathcal{C}_{3,N}$ , induced from the discrete version of the first Adler-Gelfand-Dickey Poisson bracket by Miura transformation. By studying the asymptotic behavior of these two Poisson structures, we have

**Proposition 1.1** [PROPOSITION 3.9 3.11] *We relate asymptotically the dual of the Poisson structures  $\{\cdot, \cdot\}_{\mathcal{C}_{2,N}}$  and  $\{\cdot, \cdot\}_{S2}$  to the Virasoro algebra.*

With these evidences, we compare these two Poisson structures, we have our main theorem.

**Theorem 1.2** *The Poisson structures  $\{\cdot, \cdot\}_{\mathcal{C}_{2,N}}$  and  $\{\cdot, \cdot\}_{S2}$  for  $\mathcal{C}_{2,N}$  are compatible.*

We calculate some examples for  $n = 3$ , the two Poisson structures  $\{\cdot, \cdot\}_{\mathcal{C}_{3,N}}$  and  $\{\cdot, \cdot\}_{S3}$  are not compatible.

In our further study, we will modify the definition of the induced Poisson bracket  $\{\cdot, \cdot\}_{\mathcal{C}_{3,N}}$  to make it work. We hope that this paper helps to understand the conjectures in general cases.

## 2 Poisson algebra on the configuration space of $N$ -twisted polygons

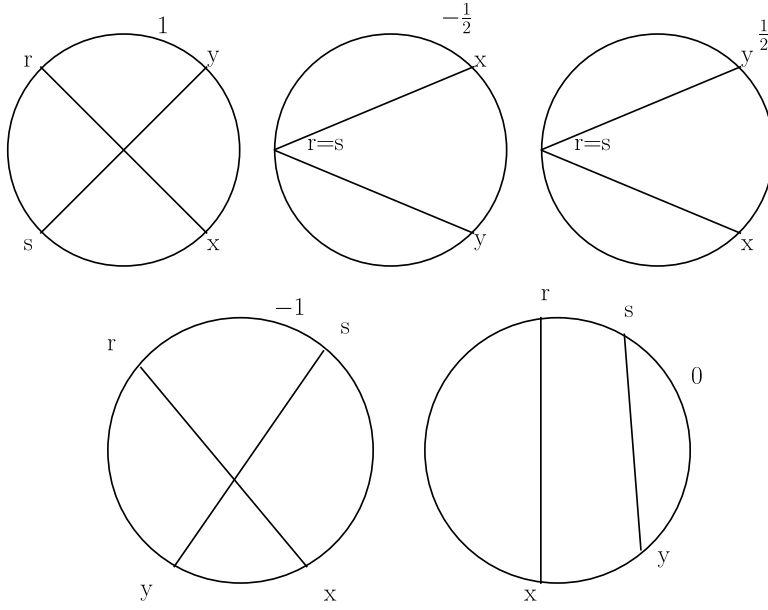
In this section, we induce a Poisson algebra on the configuration space of  $N$ -twisted polygons in  $\mathbb{RP}^{n-1}$  from the swapping algebra. We give explicit formulas for  $n = 2, 3$ . For  $n = 2$ , the induced Poisson algebra coincides with Takhtajan-Faddeev-Volkov algebra. For  $n = 3$ , the induced Poisson algebra coincides with a lattice  $W_3$  algebra.

### 2.1 Swapping algebra revisited

In this subsection, we recall briefly some definitions about the swapping algebra introduced by F. Labourie. Our definitions here are based on Section 2 of [L12].

**Definition 2.1** [LINKING NUMBER] Let  $(r, x, s, y)$  be a quadruple of four points in  $S^1$ . The *linking number* between  $rx$  and  $sy$  is

$$\mathcal{J}(rx, sy) = \frac{1}{2} \cdot (\sigma(r-x) \cdot \sigma(r-y) \cdot \sigma(y-x) - \sigma(r-x) \cdot \sigma(r-s) \cdot \sigma(s-x)), \quad (1)$$



**Fig. 1** Linking number between  $rx$  and  $sy$ .

such that for any  $a \in \mathbb{R}$ , we define  $\sigma(a)$  as follows. Remove any point  $o$  different from  $r, x, s, y$  in  $S^1$  in order to get an interval  $]0, 1[$ . Then the points  $r, x, s, y \in S^1$  correspond to the real numbers in  $]0, 1[$ ,  $\sigma(a) = -1; 0; 1$  whenever  $a < 0; a = 0; a > 0$  respectively.

In fact, the value of  $\mathcal{J}(rx, sy)$  belongs to  $\{0, \pm 1, \pm \frac{1}{2}\}$ , depends on the corresponding positions of  $r, x, s, y$ , and does not depend on the choice of the point  $o$ . In Figure 1, we describe five possible values of  $\mathcal{J}(rx, sy)$ .

Let  $\mathcal{P}$  be a finite subset of the circle  $S^1$  provided with cyclic order.  $\mathbb{K}$  is a characteristic zero field. We represent an ordered pair  $(r, x)$  of  $\mathcal{P}$  by the expression  $rx$ .

**Definition 2.2** [SWAPPING RING OF  $\mathcal{P}$ ] *The swapping ring of  $\mathcal{P}$  is the ring*

$$\mathcal{Z}(\mathcal{P}) := \mathbb{K}[\{xy\}_{\forall x, y \in \mathcal{P}}] / \{xx \mid \forall x \in \mathcal{P}\}$$

over  $\mathbb{K}$ , where  $\{xy\}_{\forall x, y \in \mathcal{P}}$  are variables with values in  $\mathbb{K}$ .

Notably,  $rx = 0$  if  $r = x$  in  $\mathcal{Z}(\mathcal{P})$ . Then we equip  $\mathcal{Z}(\mathcal{P})$  with a Poisson bracket defined by F. Labourie in Section 2 of [L12].

**Definition 2.3** [SWAPPING BRACKET] *The swapping bracket over  $\mathcal{Z}(\mathcal{P})$  is defined by extending the following formula on generators to  $\mathcal{Z}(\mathcal{P})$  by Leibniz's rule:*

$$\{rx, sy\} = \mathcal{J}(rx, sy) \cdot ry \cdot sx. \quad (2)$$

(Here is the case for  $\alpha = 0$  in Section 2 of [L12].)

**Theorem 2.4** [F. LABOURIE [L12]] *The swapping bracket as above verifies the Jacobi identity. So the swapping bracket defines a Poisson structure on  $\mathcal{Z}(\mathcal{P})$ .*

**Definition 2.5** [SWAPPING ALGEBRA OF  $\mathcal{P}$ ] The swapping algebra of  $\mathcal{P}$  is  $\mathcal{Z}(\mathcal{P})$  equipped with the swapping bracket.

**Definition 2.6** [SWAPPING FRACTION ALGEBRA OF  $\mathcal{P}$ ] The swapping fraction algebra of  $\mathcal{P}$  is fraction ring  $\mathcal{Q}(\mathcal{P})$  of  $\mathcal{Z}(\mathcal{P})$  equipped with the swapping bracket.

**Definition 2.7** [CROSS FRACTION] Let  $x, y, z, t$  belong to  $\mathcal{P}$  so that  $x \neq t$  and  $y \neq z$ . The cross fraction determined by  $(x, y, z, t)$  is the element of  $\mathcal{Q}(\mathcal{P})$ :

$$[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}. \quad (3)$$

Let  $\mathcal{CR}(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P}, x \neq t, y \neq z\}$  be the set of all the cross-fractions in  $\mathcal{Q}(\mathcal{P})$ .

**Definition 2.8** [SWAPPING MULTIFRACTION ALGEBRA OF  $\mathcal{P}$ ] Let  $\mathcal{B}(\mathcal{P})$  be the subring of  $\mathcal{Q}(\mathcal{P})$  generated by  $\mathcal{CR}(\mathcal{P})$ . The swapping multifraction algebra of  $\mathcal{P}$  is  $\mathcal{B}(\mathcal{P})$  equipped with the swapping bracket.

## 2.2 Induced Poisson structures on the configuration spaces

**Definition 2.9** [CONFIGURATION SPACE  $\mathcal{C}_n$ ] Let  $f$  be a map from  $\mathbb{Z}$  to  $\mathbb{RP}^{n-1}$  such that for any  $i_1 < \dots < i_n$  in  $\mathbb{Z}$ , the images  $f(i_1), \dots, f(i_n)$  are in general position in  $\mathbb{RP}^{n-1}$ . The configuration space  $\mathcal{C}_n$  is the space of all these maps up to projective transformations.

**Definition 2.10** [CONFIGURATION SPACE OF  $N$ -TWISTED POLYGONS IN  $\mathbb{RP}^{n-1}$ ] For  $N > n$ , a  $N$ -twisted polygon in  $\mathbb{RP}^{n-1}$  is a map  $f$  from  $\mathbb{Z}$  to  $\mathbb{RP}^{n-1}$  such that for any  $k \in \mathbb{Z}$ , we have  $f(k+N) = M_f \cdot f(k)$  where the monodromy  $M_f$  belongs to  $\text{PSL}(n, \mathbb{R})$ . We say that  $f$  is in general position if for any  $1 \leq i_1 < \dots < i_n \leq N$ , the images  $f(i_1), \dots, f(i_n)$  in  $\mathbb{RP}^{n-1}$  are in general position. The configuration space of  $N$ -twisted polygons in  $\mathbb{RP}^{n-1}$ , denoted by  $\mathcal{C}_{n,N}$ , is the space of the  $N$ -twisted polygons in general position in  $\mathbb{RP}^{n-1}$  up to projective transformations.

**Definition 2.11** [INDUCED POISSON STRUCTURE FOR  $\mathcal{C}_n$ ] Without loss of ambiguity, let  $\mathcal{P} = \{\dots, r_1, r_2, \dots\}$ , the swapping multifraction algebra of  $\mathcal{P}$  is  $(\mathcal{B}(\mathcal{P}), \{\cdot, \cdot\})$ . For each configuration  $f$  in general position, we associate  $f$  to a map  $f_{n-1}$ , where  $f_{n-1}(r_k) = f(r_k) \wedge \dots \wedge f(r_{k+n-2}) \in \mathbb{RP}^{n-1*}$  for any  $k \in \mathbb{Z}$ . We lift  $f$  ( $f_{n-1}$  resp.) to the map  $\tilde{f}$  ( $\tilde{f}_{n-1}$  resp.) with images in  $\mathbb{R}^n$  ( $\mathbb{R}^{n*}$  resp.). Let  $\Omega$  be a volume form of  $\mathbb{R}^n$ . Let  $\mathcal{B}'(\mathcal{P})$  be a subfraction ring of  $\mathcal{Q}(\mathcal{P})$  generated by  $\mathcal{CR}'(\mathcal{P})$ , where  $\mathcal{CR}'(\mathcal{P})$  is the set of the cross fractions such that for each factor  $ab$  of the cross fraction we have  $\tilde{f}(a) \wedge \tilde{f}_{n-1}(b) \neq 0$ . We define a homomorphism  $\theta$  from  $\mathcal{B}'(\mathcal{P})$  to  $C^\infty(\mathcal{C}_n)$  by extending the following formula on generators:

$$\theta\left(\frac{ac}{ad} \cdot \frac{bd}{bc}\right) = \frac{\Omega\left(\tilde{f}(a) \wedge \tilde{f}_{n-1}(c)\right)}{\Omega\left(\tilde{f}(a) \wedge \tilde{f}_{n-1}(d)\right)} \cdot \frac{\Omega\left(\tilde{f}(b) \wedge \tilde{f}_{n-1}(d)\right)}{\Omega\left(\tilde{f}(b) \wedge \tilde{f}_{n-1}(c)\right)},$$

which does not depend on the lifts and the volume form  $\Omega$ .

Then the induced Poisson bracket  $\{\cdot, \cdot\}_{\mathcal{C}_n}$  on  $\theta(\mathcal{B}'(\mathcal{P}))$  is

$$\{\theta(e), \theta(f)\}_{\mathcal{C}_n} = \theta(\{e, f\}),$$

for any  $e, f \in \mathcal{B}'(\mathcal{P})$ .

- Remark 2.12** 1. We can modify  $f_{n-1}$  such that  $f_{n-1}(r_k) = f(r_k) \wedge f(r_{k+j_1}) \wedge \dots \wedge f(r_{k+j_{n-2}}) \in \mathbb{RP}^{n-1*}$ , where  $j_1 < \dots < j_{n-2}$  in  $\mathbb{Z}$ , to get different induced Poisson brackets.
2. The kernel of  $\theta \circ \psi$  is characterized by the  $(n+1) \times (n+1)$  determinant relations in [Sul6].

**Definition 2.13** [INDUCED POISSON STRUCTURE FOR  $\mathcal{C}_{n,N}$ ] The natural embedding  $\psi$  from  $\mathcal{C}_{n,N}$  to  $\mathcal{C}_n$  induces a map  $\psi$  from  $C^\infty(\mathcal{C}_n)$  to  $C^\infty(\mathcal{C}_{n,N})$ , the induced Poisson bracket  $\{\cdot, \cdot\}_{\mathcal{C}_{n,N}}$  on  $\psi(\theta(\mathcal{B}'(\mathcal{P})))$  is

$$\{\psi(g), \psi(h)\}_{\mathcal{C}_{n,N}} = \psi(\{g, h\}_{\mathcal{C}_n}),$$

for any  $g, h \in \theta(\mathcal{B}'(\mathcal{P}))$ .

### 2.2.1 Case $n=2$

**Notation 2.1** Let

$$[a, b, c, d] := \frac{a-c}{a-d} \cdot \frac{b-d}{b-c}, \quad (4)$$

$$[x, y] := \frac{\{x, y\}}{xy}. \quad (5)$$

**Definition 2.14** [COORDINATE SYSTEM OF  $\mathcal{C}_{2,N}$ ] For any  $f \in \mathcal{C}_{2,N}$ , suppose  $f(k) = [f_k : 1] \in \mathbb{RP}^1$ , let

$$B_k = [f_{k-1}, f_{k+2}, f_{k+1}, f_k]. \quad (6)$$

We have  $\{B_k\}_{k=1}^N$  is a coordinate system of  $\mathcal{C}_{2,N}$ .

**Proposition 2.15** [FORMULAS FOR  $\{\cdot, \cdot\}_{\mathcal{C}_{2,N}}$ ] We have

$$\{B_k, B_{k+1}\}_{\mathcal{C}_{2,N}} = \left(1 - \frac{1}{B_k} - \frac{1}{B_{k+1}}\right) B_k \cdot B_{k+1}, \quad (7)$$

$$\{B_k, B_{k-1}\}_{\mathcal{C}_{2,N}} = -\left(1 - \frac{1}{B_k} - \frac{1}{B_{k-1}}\right) B_k \cdot B_{k-1}, \quad (8)$$

$$\{B_k, B_{k+2}\}_{\mathcal{C}_{2,N}} = -\frac{1}{B_{k+1}} \cdot B_k \cdot B_{k+2}, \quad (9)$$

$$\{B_k, B_{k-2}\}_{\mathcal{C}_{2,N}} = \frac{1}{B_{k-1}} \cdot B_k \cdot B_{k-2}, \quad (10)$$

for  $k = 1, \dots, N$  with the convention  $k+N = k$ . For the other cases

$$\{B_i, B_j\}_{\mathcal{C}_{2,N}} = 0. \quad (11)$$

*Proof* By calculating the swapping bracket, we have the following non-trivial equations

$$\begin{aligned} & \left[ \frac{r_{k-1}r_{k+1}}{r_{k-1}r_k} \cdot \frac{r_{k+2}r_k}{r_{k+2}r_{k+1}}, \frac{r_k r_{k+2}}{r_k r_{k+1}} \cdot \frac{r_{k+3}r_{k+1}}{r_{k+3}r_{k+2}} \right] \\ &= -1 + \frac{r_{k-1}r_{k+2} \cdot r_k r_{k+1}}{r_{k-1}r_{k+1} \cdot r_k r_{k+2}} + \frac{r_{k+2}r_{k+1} \cdot r_{k+3}r_k}{r_{k+2}r_k \cdot r_{k+3}r_{k+1}}, \end{aligned} \quad (12)$$

$$\begin{aligned}
& \left[ \frac{r_{k-1}r_{k+1}}{r_{k-1}r_k} \cdot \frac{r_{k+2}r_k}{r_{k+2}r_{k+1}}, \frac{r_{k+1}r_{k+3}}{r_{k+1}r_{k+2}} \cdot \frac{r_{k+4}r_{k+2}}{r_{k+4}r_{k+3}} \right] \\
&= -\frac{r_{k+2}r_{k+3} \cdot r_{k+1}r_k}{r_{k+2}r_k \cdot r_{k+1}r_{k+3}}.
\end{aligned} \tag{13}$$

Since

$$\begin{aligned}
& \psi \left( \theta \left( \frac{r_{k-1}r_{k+1}}{r_{k-1}r_k} \cdot \frac{r_{k+2}r_k}{r_{k+2}r_{k+1}} \right) \right) = B_k, \\
& \psi \left( \theta \left( \frac{r_{k-1}r_{k+2} \cdot r_k r_{k+1}}{r_{k-1}r_{k+1} \cdot r_k r_{k+2}} \right) \right) = 1 - \frac{1}{B_k}, \\
& \psi \left( \theta \left( \frac{r_{k+2}r_{k+1} \cdot r_{k+3}r_k}{r_{k+2}r_k \cdot r_{k+3}r_{k+1}} \right) \right) = 1 - \frac{1}{B_{k+1}}, \\
& \psi \left( \theta \left( \frac{r_{k+2}r_{k+3} \cdot r_{k+1}r_k}{r_{k+2}r_k \cdot r_{k+1}r_{k+3}} \right) \right) = \frac{1}{B_{k+1}},
\end{aligned}$$

we conclude that we verify the formulas in the proposition.  $\square$

*Remark 2.16* Faddeev-Takhtajan-Volkov algebra coincides with  $\{\cdot, \cdot\}_{\mathcal{C}_{2,N}}$ , but the original paper [FT86] works on the coordinate system  $\{s_i = \frac{1}{B_i}\}_{i=1}^N$  of  $\mathcal{C}_{2,N}$ .

### 2.2.2 Case $n=3$

**Definition 2.17** [COORDINATE SYSTEM OF  $\mathcal{C}_{3,N}$  [SOT10]] For any  $f \in \mathcal{C}_{3,N}$ , let

$$X_k = [f(k-2), f(k-2)f(k-1) \wedge f(k+1)f(k+2), f(k-1), f(k-2)f(k-1) \wedge f(k)f(k+1)] \tag{14}$$

$$Y_k = [f(k+1)f(k+2) \wedge f(k-2)f(k-1), f(k+2), f(k+1)f(k+2) \wedge f(k-1)f(k), f(k+1)]. \tag{15}$$

By R. Schwartz et al. [SOT10],  $\{X_k, Y_k\}_{k=1}^N$  is a coordinate system of  $\mathcal{C}_{3,N}$ .

**Proposition 2.18** [FORMULAS FOR  $\{\cdot, \cdot\}_{\mathcal{C}_{3,N}}$ ] We have

$$\{X_k, X_{k+1}\}_{\mathcal{C}_{3,N}} = X_k \cdot X_{k+1} \cdot (1 - Y_k) \cdot (1 - X_k - X_{k+1}), \tag{16}$$

$$\{X_k, X_{k+2}\}_{\mathcal{C}_{3,N}} = X_k \cdot X_{k+1} \cdot X_{k+2} \cdot (Y_k + Y_{k+1} - 1), \tag{17}$$

$$\{X_k, Y_{k-3}\}_{\mathcal{C}_{3,N}} = X_{k-1} \cdot X_k \cdot Y_{k-3} \cdot Y_{k-2}, \tag{18}$$

$$\{X_k, Y_{k-2}\}_{\mathcal{C}_{3,N}} = X_k \cdot Y_{k-2} \cdot (-X_{k-1} - Y_{k-1} + X_k Y_{k-1} + X_{k-1} Y_{k-1} + X_{k-1} Y_{k-2}), \tag{19}$$

$$\{X_k, Y_{k-1}\}_{\mathcal{C}_{3,N}} = X_k \cdot Y_{k-1} \cdot (1 - X_k) \cdot (1 - Y_{k-1}), \tag{20}$$

$$\{X_k, Y_k\}_{\mathcal{C}_{3,N}} = -X_k \cdot Y_k \cdot (1 - X_k) \cdot (1 - Y_k), \tag{21}$$

$$\{X_k, Y_{k+1}\}_{\mathcal{C}_{3,N}} = X_k \cdot Y_{k+1} \cdot (Y_k + X_{k+1} - X_k Y_k - X_{k+1} Y_k - X_{k+1} Y_{k+1}), \tag{22}$$

$$\{X_k, Y_{k+2}\}_{\mathcal{C}_{3,N}} = -X_k \cdot X_{k+1} \cdot Y_{k+1} \cdot Y_{k+2}, \tag{23}$$

$$\{Y_k, Y_{k+1}\}_{\mathcal{C}_{3,N}} = Y_k \cdot Y_{k+1} \cdot (1 - X_{k+1}) \cdot (1 - Y_k - Y_{k+1}), \tag{24}$$

$$\{Y_k, Y_{k+2}\}_{\mathcal{C}_{3,N}} = Y_k \cdot Y_{k+1} \cdot Y_{k+2} (X_{k+1} + X_{k+2} - 1), \tag{25}$$

for  $k = 1, \dots, N$  with the convention  $k+N = k$ . Except the above brackets and their symmetry ones by  $\{a, b\}_{\mathcal{C}_{3,N}} = -\{b, a\}_{\mathcal{C}_{3,N}}$ , all the other brackets between two generators are zero.

*Proof* By observing that

$$\psi \left( \theta \left( \frac{(k-2)(k)}{(k-2)(k+1)} \cdot \frac{(k-1)(k+1)}{(k-1)(k)} \right) \right) = \frac{1}{1-X_k}, \quad (26)$$

$$\psi \left( \theta \left( \frac{(k+2)(k-1)}{(k+2)(k-2)} \cdot \frac{(k+1)(k-2)}{(k+1)(k-1)} \right) \right) = \frac{1}{1-Y_k}. \quad (27)$$

We calculate explicitly the swapping brackets between the  $2N$  terms above. By more complicated calculations and the same argument as in Proposition 2.15, we have the above formulas.  $\square$

*Remark 2.19* We notice that the above formulas coincide with Equations (2.2.5) of [P94] when  $\tau_{2k-1}$  is replaced by  $X_k$  and  $\tau_{2k}$  is replaced by  $Y_k$ , where  $\{\tau_k\}$  are the coordinates for the lattice  $W_3$  algebra arising from the study of the quantum invariant ring.

### 3 Large $N$ asymptotic relations

There are two Poisson algebras on  $\mathcal{C}_{2,N}$ , one is the Faddeev-Takhtajan-Volkov algebra, another is induced from the discrete version of the first Adler-Gelfand-Dickey Poisson bracket by Miura transformation. In this section, we relate asymptotically the dual of these Poisson algebras to the Virasoro algebra.

#### 3.1 The discrete Hill's operator and the cross-ratios

**Definition 3.1** [DISCRETE HILL'S EQUATION] Let  $N \geq 1$  be an integer. Given a periodic sequence  $\{H_k\}_{k=-\infty}^{\infty}$  in  $\mathbb{R}$  where  $H_{N+k} = H_k$  for any  $k \in \mathbb{Z}$ . The *discrete Hill's equation* is the difference equation in  $\{C_k\}_{k=-\infty}^{\infty}$ :

$$\frac{\frac{C_{k+1}-C_k}{N} - \frac{C_k-C_{k-1}}{N}}{N} = H_k \cdot C_k, \quad (28)$$

or equivalently

$$C_{k+1} = \left( \frac{H_k}{N^2} + 2 \right) \cdot C_k - C_{k-1}, \quad (29)$$

for any  $k$  belongs to  $\mathbb{Z}$ .

We define  $\{H_k\}_{k=-\infty}^{\infty}$  to be a *discrete Hill's operator*, and  $\{C_k\}_{k=-\infty}^{\infty}$  is the *solution* to the discrete Hill's equation.

Given a discrete Hill's operator, by Equation 29, the series  $\{C_k\}_{k=-\infty}^{\infty}$  is fixed when two initial values  $C_0, C_1$  are given. Since the difference equation is homogeneous, so up to scalar and  $\text{PSL}(2, \mathbb{R})$  projective transformation, there are exactly two linear independent solutions of the discrete Hill's operator, which are corresponding to one point of  $\mathcal{C}_{2,N}$ .

Conversely, when  $N$  is odd, given a point of  $\mathcal{C}_{2,N}$ , there is one unique Hill's operator corresponding to it by the similar argument of Proposition 4.1 of [SOT10].

**Proposition 3.2** [[SOT10]] Let  $N > 3$  be odd, let  $\{f(i)\}_{i \in \mathbb{Z}} \in \mathcal{C}_{2,N}$ ,  $f(i) = [f_i : 1] \in \mathbb{RP}^1$ . There exists one unique discrete Hill's equation 29 such that  $\{X_i\}_{i=-\infty}^{\infty}$  and  $\{Y_i\}_{i=-\infty}^{\infty}$  are two linear independent solutions of 29 and  $[X_i : Y_i] = [f_i : 1]$ .

**Notation 3.1** Let  $b_k = \frac{H_k}{N^2} + 2$ .

**Corollary 3.3** Let  $N > 3$  be odd,  $\{b_k\}_{k=1}^N$  and  $\{H_k\}_{k=1}^N$  are two coordinate systems of  $\mathcal{C}_{2,N}$ .

The following proposition explains the relation between the discrete Hill's operator and cross ratio coordinate  $\{B_k\}_{k=1}^N$ .

**Proposition 3.4** Let  $\{X_i\}_{i=-\infty}^\infty$  and  $\{Y_i\}_{i=-\infty}^\infty$  be two linear independent solutions of 29 and  $[X_i : Y_i] = [f_i : 1]$ . For any  $k \in \mathbb{Z}$ , we have

$$B_k = [f_{k-1}, f_{k+2}, f_{k+1}, f_k] = b_k \cdot b_{k+1}. \quad (30)$$

*Proof* Since  $\{X_i\}_{i=-\infty}^\infty$  and  $\{Y_i\}_{i=-\infty}^\infty$  are linear independent, we have

$$h := X_1 Y_0 - X_0 Y_1 \neq 0.$$

For any  $k \geq 0$ , we have

$$\begin{aligned} X_{k+1} Y_k - X_k Y_{k+1} &= (b_k X_k - X_{k-1}) \cdot Y_k - X_k \cdot (b_k Y_k - Y_{k-1}) = X_k Y_{k-1} - X_{k-1} Y_k \\ &= \dots = X_1 Y_0 - X_0 Y_1 = h, \end{aligned} \quad (31)$$

$$f_{k+1} - f_k = \frac{X_{k+1}}{Y_{k+1}} - \frac{X_k}{Y_k} = \frac{h}{Y_k Y_{k+1}}, \quad (32)$$

$$\begin{aligned} f_{k+2} - f_k &= \frac{X_{k+2}}{Y_{k+2}} - \frac{X_k}{Y_k} = \frac{X_{k+2} Y_k - X_k Y_{k+2}}{Y_k Y_{k+2}} \\ &= \frac{(b_{k+1} X_{k+1} - X_k) \cdot Y_k - X_k \cdot (b_{k+1} Y_{k+1} - Y_k)}{Y_k Y_{k+2}} = \frac{h b_{k+1}}{Y_k Y_{k+1}}. \end{aligned} \quad (33)$$

Thus we have

$$\frac{f_{k-1} - f_{k+1}}{f_{k-1} - f_k} \cdot \frac{f_{k+2} - f_k}{f_{k+2} - f_{k+1}} = \frac{-\frac{h b_k}{Y_{k-1} Y_{k+1}}}{-\frac{h}{Y_{k-1} Y_k}} \cdot \frac{\frac{h b_{k+1}}{Y_{k+2} Y_k}}{\frac{h}{Y_{k+2} Y_{k+1}}} = b_k b_{k+1} \quad (34)$$

for any  $k \geq 1$ .

By the similar argument, for  $k < 1$ , we have  $B_k = b_k \cdot b_{k+1}$ .

We conclude that  $B_k = b_k \cdot b_{k+1}$  for any  $k \in \mathbb{Z}$ .  $\square$

**Remark 3.5** The cross ratio  $B_k$  can be understood as Schwarzian derivative [OT05].

### 3.2 Virasoro algebra and $\{\cdot, \cdot\}_{\mathcal{C}_{2,N}}$

By comparing with Proposition 2.3 of [KW09] page 67, we define a discrete version of Virasoro algebra.

**Definition 3.6**  $[(t_1, t_2, N)\text{-VIRASORO BRACKET}]$  Let  $t_1, t_2 \in \mathbb{R}$  and  $N \in \mathbb{N}$ , the  $(t_1, t_2, N)$ -Virasoro bracket on  $\{I_k\}_{k=-N}^N$  is defined to be:

For  $p, q = -\lfloor \frac{N-1}{2} \rfloor, \dots, \lfloor \frac{N}{2} \rfloor$ ,

1. when  $p \neq -q$ , we have

$$\{I_p, I_q\}_{N, t_1, t_2} = (p - q) \cdot I_{p+q}$$

with the convention  $I_{k+N} = I_k$ ;



2. when  $p = -q$ , we have

$$\{I_p, I_{-p}\}_{N, t_1, t_2} = 2p \cdot I_0 + t_1 \cdot p^3 + t_2 \cdot p.$$

*Remark 3.7* Notice that  $(t_1, t_2, N)$ -Virasoro bracket is asymptotic to the Poisson bracket associated to the 2-cocycle with  $c_1 = t_1$ ,  $c_2 = t_2$  as in Proposition 2.3 of [KW09] page 67 when  $N$  converges to infinite, but it is not a Poisson bracket.

Very specific values of  $t_1$  and  $t_2$  correspond to Virasoro algebra. When  $t_1$  is fixed,  $t_2$  varies, they correspond to same element in the cohomology group  $H^2(\text{Vect}(S^1), \mathbb{R})$ . Different  $t_1$  corresponds to different element in the one dimensional space  $H^2(\text{Vect}(S^1), \mathbb{R})$ .

**Definition 3.8** [DISCRETE FOURIER TRANSFORMATION] Let  $\{B_k\}_{k=1}^N$  be the cross ratio coordinates of  $\mathcal{C}_{2,N}$ . Let  $\mathbb{B} = \{B_1, \dots, B_N\}$ . The *discrete Fourier transformation*  $\mathcal{F}$  of  $\mathbb{B}$  is defined to be

$$\mathcal{F}_p \mathbb{B} = \sum_{k=1}^N B_k e^{-\frac{2pk\pi i}{N}}. \quad (35)$$

Our main proposition of this subsection is

**Proposition 3.9** [LARGE N ASYMPTOTIC] Let  $N > 3$ . For  $k = -\lfloor \frac{N-1}{2} \rfloor, \dots, \lfloor \frac{N}{2} \rfloor$ , let

$$V_k = \frac{\mathcal{F}_k \mathbb{B} \cdot N}{8\pi i}.$$

We have

$$\{V_p, V_q\}_{\mathcal{C}_{2,N}} = \{V_p, V_q\}_{N, \frac{8\pi^2}{N}, 8N} + o\left(\frac{1}{N^2}\right). \quad (36)$$

*Proof* For  $p, q = -\lfloor \frac{N-1}{2} \rfloor, \dots, \lfloor \frac{N}{2} \rfloor$ , we have

$$\begin{aligned} & \{\mathcal{F}_p \mathbb{B}, \mathcal{F}_q \mathbb{B}\}_{\mathcal{C}_{2,N}} \\ &= \sum_{k=1}^N \left( e^{-\frac{2pk\pi i}{N}} \cdot e^{-\frac{2q(k+1)\pi i}{N}} - e^{-\frac{2p(k+1)\pi i}{N}} \cdot e^{-\frac{2qk\pi i}{N}} \right) \cdot (B_k B_{k+1} - B_k - B_{k+1}) \\ & \quad - \left( e^{-\frac{2pk\pi i}{N}} \cdot e^{-\frac{2q(k+2)\pi i}{N}} - e^{-\frac{2p(k+2)\pi i}{N}} \cdot e^{-\frac{2qk\pi i}{N}} \right) \cdot \frac{B_k B_{k+2}}{B_{k+1}} \end{aligned} \quad (37)$$

By

$$B_k = b_k b_{k+1} = 4 + \frac{2(H_k + H_{k+1})}{N^2} + \frac{H_k H_{k+1}}{N^4},$$

we have

$$B_k = 4 + \frac{4H_k}{N^2} + o\left(\frac{1}{N^2}\right).$$

We have the above formula equals to

$$\begin{aligned} & \sum_{k=1}^N e^{-\frac{2(p+q)k\pi i}{N}} \cdot \left[ \left( \left( 1 + \frac{-2q\pi i}{N} + \frac{-2\pi^2 q^2}{N^2} + \frac{4\pi^3 q^3 i}{3N^3} + o\left(\frac{1}{N^3}\right) \right) - \left( 1 + \frac{-2p\pi i}{N} + \frac{-2\pi^2 p^2}{N^2} + \frac{4\pi^3 p^3 i}{3N^3} + o\left(\frac{1}{N^3}\right) \right) \right) \cdot \left( 8 + \frac{24H_k}{N^2} + o\left(\frac{1}{N^2}\right) \right) - \left( \left( 1 + \frac{-4q\pi i}{N} + \frac{-8\pi^2 q^2}{N^2} + \frac{32\pi^3 q^3 i}{3N^3} + o\left(\frac{1}{N^3}\right) \right) - \left( 1 + \frac{-4p\pi i}{N} + \frac{-8\pi^2 p^2}{N^2} + \frac{32\pi^3 p^3 i}{3N^3} + o\left(\frac{1}{N^3}\right) \right) \right) \cdot \left( 4 + \frac{4H_k}{N^2} + o\left(\frac{1}{N^2}\right) \right) \right] \\ &= \sum_{k=1}^N e^{-\frac{2(p+q)k\pi i}{N}} \cdot \left[ \frac{32H_k(p-q)\pi i}{N^3} - \frac{16\pi^2(p^2-q^2)}{N^2} + \frac{32\pi^3(p^3-q^3)i}{N^3} + o\left(\frac{1}{N^3}\right) \right]. \end{aligned} \quad (38)$$

When  $p \neq -q$ , since  $\sum_{k=1}^N e^{\frac{-2(p+q)k\pi i}{N}} = 0$ , by  $B_k = 4 + \frac{4H_k}{N^2} + o(\frac{1}{N^2})$ , the above formula equals to

$$\sum_{k=1}^N e^{\frac{-2(p+q)k\pi i}{N}} \cdot \frac{8B_k(p-q)\pi i}{N} = \frac{8(p-q)\pi i}{N} \cdot \mathcal{F}_{p+q}\mathbb{B} + o\left(\frac{1}{N^3}\right); \quad (39)$$

When  $p = -q$ , the above formula equals to

$$\begin{aligned} & \frac{64p\pi i}{N^3} \sum_{k=1}^N H_k + \frac{64p^3\pi^3 i}{N^2} + o\left(\frac{1}{N^3}\right) \\ &= \frac{16p\pi i}{N} \sum_{k=1}^N (B_k - 4) + \frac{64p^3\pi^3 i}{N^2} + o\left(\frac{1}{N^3}\right) \\ &= \frac{16p\pi i}{N} \mathcal{F}_0\mathbb{B} - 64p\pi i + \frac{64p^3\pi^3 i}{N^2} + o\left(\frac{1}{N^3}\right). \end{aligned} \quad (40)$$

Replacing  $\mathcal{F}_k$  by

$$V_k = \frac{\mathcal{F}_k\mathbb{B} \cdot N}{8\pi i},$$

we obtain that:

for  $p \neq -q$ ,

$$\{V_p, V_q\}_{\mathcal{C}_{2,N}} = (p-q) \cdot V_{p+q} + o\left(\frac{1}{N^2}\right);$$

for  $p = -q$

$$\{V_p, V_{-p}\}_{\mathcal{C}_{2,N}} = 2p \cdot V_0 + \left(\frac{8\pi^2}{N}\right) \cdot p^3 - 8N \cdot p + o\left(\frac{1}{N^2}\right).$$

We conclude that

$$\{V_p, V_q\}_{\mathcal{C}_{2,N}} = \{V_p, V_q\}_{N, \frac{8\pi^2}{N}, 8N} + o\left(\frac{1}{N^2}\right). \quad (41)$$

□

### 3.3 Poisson structure via Miura transformation and its large N asymptotic

The discrete version of the Miura transformation  $\mu$  [V88] [FRS98]:

$$\mu(v_k) = B_k = \frac{1}{s_k} = (1 + v_k)(1 + v_{k+1}^{-1}) \quad (42)$$

is a Poisson map with respect to the Poisson bracket:

$$\{v_i, v_j\}_{V2} = (\delta_{i+1,j} - \delta_{i-1,j}) \cdot v_i \cdot v_j \quad (43)$$

corresponding to the first Adler-Gelfand-Dickey Poisson bracket.

**Definition 3.10** [ANOTHER POISSON BRACKET FOR  $\mathcal{C}_{2,N}$ ] Let us consider the Miura transformation as a map  $\mu$  from  $\mathbb{R}(v_1, \dots, v_N)$  to  $\mathbb{R}(B_1, \dots, B_N)$ . The map  $\mu$  induce another Poisson bracket for  $\mathcal{C}_{2,N}$  by:

$$\{B_i, B_j\}_{S2} = \{\mu(v_i), \mu(v_j)\}_{S2} := \mu\{v_i, v_j\}_{V2}.$$

Thus we have

$$\{B_i, B_j\}_{S2} = (\delta_{i+1,j} - \delta_{i-1,j}) \cdot B_i \cdot B_j. \quad (44)$$

We have similar result as Proposition 3.9 for  $\{\cdot, \cdot\}_{S2}$ .

**Proposition 3.11** [LARGE N ASYMPTOTIC] Let  $N > 3$ . For  $k = -\lfloor \frac{N-1}{2} \rfloor, \dots, \lfloor \frac{N}{2} \rfloor$ , let

$$W_k = \frac{\mathcal{F}_k \mathbb{B} \cdot N}{16\pi i}.$$

We have

$$\{W_p, W_q\}_{S2} = \{W_p, W_q\}_{N, \frac{8\pi^2}{3N}, 4-8N} + o\left(\frac{1}{N^2}\right). \quad (45)$$

*Proof* For  $p, q = -\lfloor \frac{N-1}{2} \rfloor, \dots, \lfloor \frac{N}{2} \rfloor$ , we have

$$\{\mathcal{F}_p \mathbb{B}, \mathcal{F}_q \mathbb{B}\}_{S2} = \sum_{k=1}^N \left( e^{\frac{-2pk\pi i}{N}} \cdot e^{\frac{-2q(k+1)\pi i}{N}} - e^{\frac{-2p(k+1)\pi i}{N}} \cdot e^{\frac{-2qk\pi i}{N}} \right) \cdot (B_k B_{k+1}) \quad (46)$$

By

$$B_k = a_k a_{k+1} = 4 + \frac{2(H_k + H_{k+1})}{N^2} + \frac{H_k H_{k+1}}{N^4},$$

we have

$$B_k = 4 + \frac{4H_k}{N^2} + o\left(\frac{1}{N^2}\right).$$

We have the above formula equals to

$$\begin{aligned} & \sum_{k=1}^N e^{\frac{-2(p+q)k\pi i}{N}} \cdot \left( \left( 1 + \frac{-2q\pi i}{N} + \frac{-2\pi^2 q^2}{N^2} + \frac{4\pi^3 q^3 i}{3N^3} + o\left(\frac{1}{N^3}\right) \right) - \left( 1 + \frac{-2p\pi i}{N} + \right. \right. \\ & \quad \left. \left. + \frac{-2\pi^2 \cdot p^2}{N^2} + \frac{4\pi^3 p^3 i}{3N^3} + o\left(\frac{1}{N^3}\right) \right) \right) \cdot \left( 16 + \frac{32H_k}{N^2} + o\left(\frac{1}{N^2}\right) \right) \\ &= \sum_{k=1}^N e^{\frac{-2(p+q)k\pi i}{N}} \cdot \left[ \frac{32\pi i(p-q)}{N} + \frac{32\pi^2(p^2-q^2)}{N^2} - \frac{64\pi^3(p^3-q^3)i}{3N^3} + \right. \\ & \quad \left. + \frac{64H_k(p-q)\pi i}{N^3} + o\left(\frac{1}{N^3}\right) \right]. \end{aligned} \quad (47)$$

When  $p \neq -q$ , since  $\sum_{k=1}^N e^{\frac{-2(p+q)k\pi i}{N}} = 0$ , by  $B_k = 4 + \frac{4H_k}{N^2} + o\left(\frac{1}{N^2}\right)$ , the above formula equals to

$$\sum_{k=1}^N e^{\frac{-2(p+q)k\pi i}{N}} \cdot \frac{16B_k(p-q)\pi i}{N} = \frac{16(p-q)\pi i}{N} \cdot \mathcal{F}_{p+q} \mathbb{B} + o\left(\frac{1}{N^3}\right); \quad (48)$$

When  $p = -q$ , since  $\sum_{k=1}^N B_k = 0$ , the above formula equals to

$$\begin{aligned} & \frac{64p\pi i}{N} + \frac{128p\pi i}{N^3} \sum_{k=1}^N H_k - \frac{128p^3\pi^3 i}{3N^2} + o\left(\frac{1}{N^3}\right) \\ &= \frac{64p\pi i}{N} + \frac{32p\pi i}{N} \sum_{k=1}^N (B_k - 4) - \frac{128p^3\pi^3 i}{3N^2} + o\left(\frac{1}{N^3}\right) \\ &= \frac{64p\pi i}{N} + \frac{32p\pi i}{N} \mathcal{F}_0 \mathbb{B} - 128p\pi i - \frac{128p^3\pi^3 i}{3N^2} + o\left(\frac{1}{N^3}\right). \end{aligned} \quad (49)$$

Thus we have:

1. for  $p \neq -q$

$$\{W_p, W_q\}_{S2} = (p - q) \cdot W_{p+q} + o\left(\frac{1}{N^2}\right);$$

2. for  $p = -q$

$$\{W_p, W_{-p}\}_{S2} = 2p \cdot W_0 + (4 - 8N) \cdot p - \frac{8\pi^2}{3N} \cdot p^3 + o\left(\frac{1}{N^2}\right).$$

We conclude that

$$\{W_p, W_q\}_{S2} = \{W_p, W_q\}_{N, \frac{8\pi^2}{3N}, 4-8N} + o\left(\frac{1}{N^2}\right). \quad (50)$$

□

#### 4 Two Poisson structures are compatible

With the evidences shown in Proposition 3.9 3.11, we prove that  $\{\cdot, \cdot\}_{\mathcal{C}_{2,N}}$  and  $\{\cdot, \cdot\}_{S2}$  are compatible in this section. Firstly, let us recall the traditional definition of the bihamiltonian system.

**Definition 4.1** [[KW09], P47, DEFINITION 4.19] Two Poisson brackets  $\{\cdot, \cdot\}_a$  and  $\{\cdot, \cdot\}_b$  for a manifold  $M$  are said to be compatible if and only if for any  $\lambda$ ,  $\{\cdot, \cdot\}_a + \lambda\{\cdot, \cdot\}_b$  is Poisson.

A dynamic system  $\frac{d}{dt}m = \xi(m)$  over  $M$  is bihamiltonian if its vector field  $\xi$  is Hamiltonian with respect to these two Poisson brackets  $\{\cdot, \cdot\}_a$  and  $\{\cdot, \cdot\}_b$ .

Following the above definition, we make small modification for a ring  $R \subset C^\infty(M, \mathbb{R})$ .

**Definition 4.2** Two Poisson brackets  $\{\cdot, \cdot\}_a$  and  $\{\cdot, \cdot\}_b$  on a ring  $R \subset C^\infty(M, \mathbb{R})$  are said to be compatible if and only if for any  $\lambda \in \mathbb{R}$ ,  $\{\cdot, \cdot\}_a + \lambda\{\cdot, \cdot\}_b$  is Poisson.

By definition, it is easy to verify that

**Proposition 4.3**  $\{\cdot, \cdot\}_a$  and  $\{\cdot, \cdot\}_b$  are compatible if and only if for any  $x, y, z \in R$ , we have

$$\begin{aligned} & \{\{x, y\}_a, z\}_b + \{\{y, z\}_a, x\}_b + \{\{z, x\}_a, y\}_b + \\ & + \{\{x, y\}_b, z\}_a + \{\{y, z\}_b, x\}_a + \{\{z, x\}_b, y\}_a = 0 \end{aligned}$$

Our main result of this paper is the following.

**Theorem 4.4** [MAIN RESULT] For  $N \geq 5$ ,  $\{\cdot, \cdot\}_{\mathcal{C}_{2,N}}$  and  $\{\cdot, \cdot\}_{S2}$  are compatible on  $\mathbb{R}(B_1, \dots, B_N)$ .

*Proof* Let

$$\begin{aligned} K(B_i, B_j, B_k) &:= \{\{B_i, B_j\}_{\mathcal{C}_{2,N}}, B_k\}_{S2} + \{\{B_j, B_k\}_{\mathcal{C}_{2,N}}, B_i\}_{S2} + \{\{B_k, B_i\}_{\mathcal{C}_{2,N}}, B_j\}_{S2} \\ &+ \{\{B_i, B_j\}_{S2}, B_k\}_{\mathcal{C}_{2,N}} + \{\{B_j, B_k\}_{S2}, B_i\}_{\mathcal{C}_{2,N}} + \{\{B_k, B_i\}_{S2}, B_j\}_{\mathcal{C}_{2,N}}. \end{aligned} \quad (51)$$

By definition, we have to check that

$$K(B_i, B_j, B_k) = 0$$

for any  $i, j, k = 1, \dots, N$ . Since

$$K(B_i, B_j, B_k) = -K(B_j, B_i, B_k),$$

when some indexes coincide, for example  $i = j$ , we have

$$K(B_i, B_i, B_k) = 0.$$

Let  $\sigma_s$  be the permutation of the  $N$  indexes such that  $\sigma(l) = l + s$ . The permutation  $\sigma_s$  induce a ring automorphism  $\chi_s$  of  $\mathbb{R}(B_1, \dots, B_N)$  such that

$$\chi_s(B_l) = B_{l+s}$$

for  $l = 1, \dots, N$ . Moreover, we have

$$\{\chi_s(B_i), \chi_s(B_j)\}_{S2} = \chi_s(\{B_i, B_j\}_{S2})$$

and

$$\{\chi_s(B_i), \chi_s(B_j)\}_{\mathcal{C}_{2,N}} = \chi_s(\{B_i, B_j\}_{\mathcal{C}_{2,N}}).$$

Let  $\tau$  be the permutation of the  $N$  indexes such that

$$\tau(l) = N + 1 - l$$

for  $l = 1, \dots, N$ . The permutation  $\tau$  induce a ring automorphism  $v$  of  $\mathbb{R}(B_1, \dots, B_N)$  such that

$$v(B_l) = B_{N+1-l}.$$

Moreover, we have

$$\{v(B_i), v(B_j)\}_{S2} = -v(\{B_i, B_j\}_{S2}),$$

$$\{v(B_i), v(B_j)\}_{\mathcal{C}_{2,N}} = -v(\{B_i, B_j\}_{\mathcal{C}_{2,N}}).$$

By the above symmetry, we suppose that

$$i = 1$$

and

$$1 < j < k \leq N.$$

Let

$$l := \min\{|j - i|, |j - i - N|, |k - j|, |k - j - N|, |i - k|, |i - k - N|\}.$$

We suppose that  $l = |j - 1|$ , we have to verify the following cases:

1. When  $1 < j - 1 < k - 2 < N - 1$ , we have

$$\begin{aligned} K(B_1, B_i, B_k) &= \{\{B_1, B_j\}_{\mathcal{C}_{2,N}}, B_k\}_{S2} + \{\{B_j, B_k\}_{\mathcal{C}_{2,N}}, B_1\}_{S2} + \{\{B_k, B_1\}_{\mathcal{C}_{2,N}}, B_j\}_{S2} \\ &+ \{\{B_1, B_j\}_{S2}, B_k\}_{\mathcal{C}_{2,N}} + \{\{B_j, B_k\}_{S2}, B_1\}_{\mathcal{C}_{2,N}} + \{\{B_k, B_1\}_{S2}, B_j\}_{\mathcal{C}_{2,N}} \\ &= \{\{B_1, B_j\}_{\mathcal{C}_{2,N}}, B_k\}_{S2} + \{\{B_j, B_k\}_{\mathcal{C}_{2,N}}, B_1\}_{S2} + \{\{B_k, B_1\}_{\mathcal{C}_{2,N}}, B_j\}_{S2}. \end{aligned} \tag{52}$$

Since

$$\{B_1, B_j\}_{\mathcal{C}_{2,N}}$$

is a polynomial of  $B_1, \dots, B_j$ , we have

$$\{\{B_1, B_j\}_{\mathcal{C}_{2,N}}, B_k\}_{S_2} = 0.$$

Similarly, we have

$$\{\{B_j, B_k\}_{\mathcal{C}_{2,N}}, B_1\}_{S_2} = 0$$

and

$$\{\{B_k, B_1\}_{\mathcal{C}_{2,N}}, B_j\}_{S_2} = 0.$$

We conclude that

$$K(B_1, B_i, B_k) = 0.$$

2. When  $j = 2, k = 3$ , we have

$$\begin{aligned} & K(B_1, B_2, B_3) \\ &= \{\{B_1, B_2\}_{\mathcal{C}_{2,N}}, B_3\}_{S_2} + \{\{B_2, B_3\}_{\mathcal{C}_{2,N}}, B_1\}_{S_2} + \{\{B_3, B_1\}_{\mathcal{C}_{2,N}}, B_2\}_{S_2} \\ &+ \{\{B_1, B_2\}_{S_2}, B_3\}_{\mathcal{C}_{2,N}} + \{\{B_2, B_3\}_{S_2}, B_1\}_{\mathcal{C}_{2,N}} + \{\{B_3, B_1\}_{S_2}, B_2\}_{\mathcal{C}_{2,N}} \\ &= \{B_1 B_2 - B_1 - B_2, B_3\}_{S_2} + \{B_2 B_3 - B_2 - B_3, B_1\}_{S_2} + \left\{\frac{B_3 B_1}{B_2}, B_2\right\}_{S_2} \\ &+ \{B_1 B_2, B_3\}_{\mathcal{C}_{2,N}} + \{B_2 B_3, B_1\}_{\mathcal{C}_{2,N}} \\ &= (B_1 - 1)B_2 B_3 - (B_3 - 1)B_1 B_2 + B_1(B_2 B_3 - B_2 - B_3) \\ &- \frac{B_1 B_3}{B_2} B_2 - (B_1 B_2 - B_1 - B_2)B_3 + \frac{B_3 B_1}{B_2} B_2 \\ &= 0. \end{aligned} \tag{53}$$

3. When  $j = 2, k = 4$  and  $N > 5$ , we have

$$\begin{aligned} & K(B_1, B_2, B_4) \\ &= \{\{B_1, B_2\}_{\mathcal{C}_{2,N}}, B_4\}_{S_2} + \{\{B_2, B_4\}_{\mathcal{C}_{2,N}}, B_1\}_{S_2} + \{\{B_4, B_1\}_{\mathcal{C}_{2,N}}, B_2\}_{S_2} \\ &+ \{\{B_1, B_2\}_{S_2}, B_4\}_{\mathcal{C}_{2,N}} + \{\{B_2, B_4\}_{S_2}, B_1\}_{\mathcal{C}_{2,N}} + \{\{B_4, B_1\}_{S_2}, B_2\}_{\mathcal{C}_{2,N}} \\ &= \{B_1 B_2 - B_1 - B_2, B_4\}_{S_2} + \left\{-\frac{B_2 B_4}{B_3}, B_1\right\}_{S_2} + \{B_1 B_2, B_4\}_{\mathcal{C}_{2,N}} \\ &= \frac{B_1 B_2 B_4}{B_3} - \frac{B_1 B_2 B_4}{B_3} \\ &= 0. \end{aligned} \tag{54}$$

4. When  $j = 2, k = 4$  and  $N = 5$ , we have

$$\begin{aligned} & K(B_1, B_2, B_4) \\ &= \{\{B_1, B_2\}_{\mathcal{C}_{2,N}}, B_4\}_{S_2} + \{\{B_2, B_4\}_{\mathcal{C}_{2,N}}, B_1\}_{S_2} + \{\{B_4, B_1\}_{\mathcal{C}_{2,N}}, B_2\}_{S_2} \\ &+ \{\{B_1, B_2\}_{S_2}, B_4\}_{\mathcal{C}_{2,N}} + \{\{B_2, B_4\}_{S_2}, B_1\}_{\mathcal{C}_{2,N}} + \{\{B_4, B_1\}_{S_2}, B_2\}_{\mathcal{C}_{2,N}} \\ &= \{B_1 B_2 - B_1 - B_2, B_4\}_{S_2} + \left\{-\frac{B_2 B_4}{B_3}, B_1\right\}_{S_2} + \left\{-\frac{B_4 B_1}{B_5}, B_2\right\}_{S_2} \\ &+ \{B_1 B_2, B_4\}_{\mathcal{C}_{2,N}} \\ &= \frac{B_1 B_2 B_4}{B_3} - \frac{B_1 B_4 B_2}{B_5} - \frac{B_1 B_2 B_4}{B_3} + \frac{B_1 B_4 B_2}{B_5} \\ &= 0. \end{aligned} \tag{55}$$

We conclude that for  $N \geq 5$ ,  $\{\cdot, \cdot\}_{\mathcal{C}_{2,N}}$  and  $\{\cdot, \cdot\}_{S_2}$  are compatible on  $\mathbb{R}(B_1, \dots, B_N)$ .  $\square$

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